Novel Equivalent Circuit for Z_{in} or Y_{in} of an Arbitrarily Terminated Transmission Line

Timothy J. Maloney
Intel Corporation

Abstract: A new equivalent circuit for Z_{in} or Y_{in} of an arbitrarily terminated transmission line is presented, one which employs only lumped impedances and short and open line stubs in series-parallel combination. The new picture readily illustrates concepts of line matching and impedance inversion at quarter-wave frequencies, and also offers insights into transient and off-resonance response.

Many books on high-frequency electrical engineering ([1, 2] are just two) solve the one-dimensional wave equation for a transmission line and give the impedance looking into a transmission line terminated by load Z_L as

$$Z_{in} = Z_0\left[\frac{Z_L\cosh(\gamma l) + Z_0\sinh(\gamma l)}{Z_0\cosh(\gamma l) + Z_L\sinh(\gamma l)}\right] \quad . \qquad (1)$$

For the customary RLGC transmission line of length ℓ, Z_0 is the characteristic impedance $Z_0 = \sqrt{\frac{R+Ls}{G+Cs}}$ and the propagation constant is $\gamma = \sqrt{(R+Ls)(G+Cs)}$; both are in terms of complex frequency $s=\sigma+j\omega$ along with circuit elements R, L, G, and C. Figure 1 shows the termination of the transmission line by load Z_L; our interest is in the one-port characteristic impedance Z_{in}, as well as its reciprocal, the admittance Y_{in}. Very often, Z_{in} (or a succession of Z_{in} solutions using successive Z_L values) is sufficient for solving a problem because we want to know the effect of the load on the source. While there are 2-port pi and T models of transmission line sections that have some relation here [3], they will not be discussed.

Generations of students have stepped through the derivation of (1) and then have wrestled with its implications, dutifully using (1) to show impedance effects at even and odd numbers of quarter-waves, and showing what value of Z_0 will match a source impedance Z_s at quarter-wave frequency, given load impedance Z_L. What apparently has not been available in the textbooks is a useful alternate picture of Fig. 1 that helps everyone to remember the essential effects.

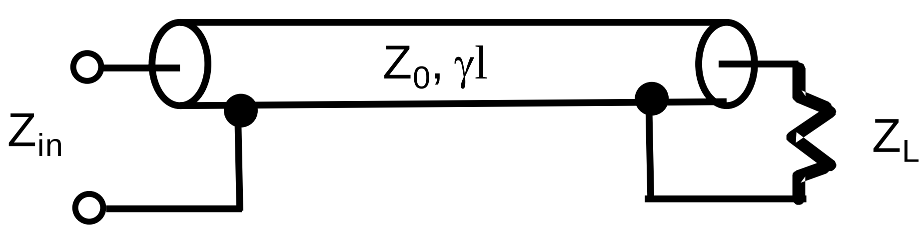

Figure 1. Transmission line Z_0 of length l and propagation constant γ terminated by load impedance Z_L; input impedance Z_{in} is desired. Z_0 and Z_L can be complex impedances $Z_L(s)$ and $Z_0(s)$.

We now use Eq. (1) to derive a useful equivalent circuit for Z_{in} that will illustrate, at a glance, some of the memorable properties of a terminated transmission line.

Equation (1), after dividing top and bottom by sinh($\gamma\ell$), can be written as a sum of two terms,

$$Z_{in} = \frac{Z_L Z_0 \coth(\gamma l)}{Z_0 \coth(\gamma l) + Z_L} + \frac{Z_0^{\ 2}}{Z_0 \coth(\gamma l) + Z_L} \quad . (2)$$

The first term can already be recognized as the parallel impedance of Z_L and Z_0coth($\gamma\ell$), to which we will return shortly. Now multiply the second term of (2) by "1" in the following way:

$$Z_{in} = \frac{Z_L Z_0 \coth(\gamma l)}{Z_0 \coth(\gamma l) + Z_L} + \frac{Z_0^{\ 2}}{Z_0 \coth(\gamma l) + Z_L} \cdot \frac{\tanh(\gamma l)}{\tanh(\gamma l)} \cdot \frac{Z_0}{Z_0} \cdot \frac{Z_L}{Z_L} \quad . (3)$$

The various new factors in the second term can be associated with top and bottom and multiplied out to give

$$Z_{in} = \frac{Z_L Z_0 \coth(\gamma l)}{Z_0 \coth(\gamma l) + Z_L} + \frac{\left[\frac{Z_0^{\ 2}}{Z_L}\right] Z_0 \tanh(\gamma l)}{\frac{Z_0^{\ 2}}{Z_L} + Z_0 \tanh(\gamma l)} \quad . (4)$$

Now both terms of Z_{in} are parallel impedances, and they add in series. This means that

$$Z_{in} = \left[\frac{Z_0^{\ 2}}{Z_L}\right] \Big\| Z_0 \tanh(\gamma l) + Z_L \,\|\, Z_0 \coth(\gamma l) \quad . (5)$$

The Z_0coth and Z_0tanh factors are well known as open and shorted stub impedances, respectively, so the new equivalent circuit is as shown in Figure 2.

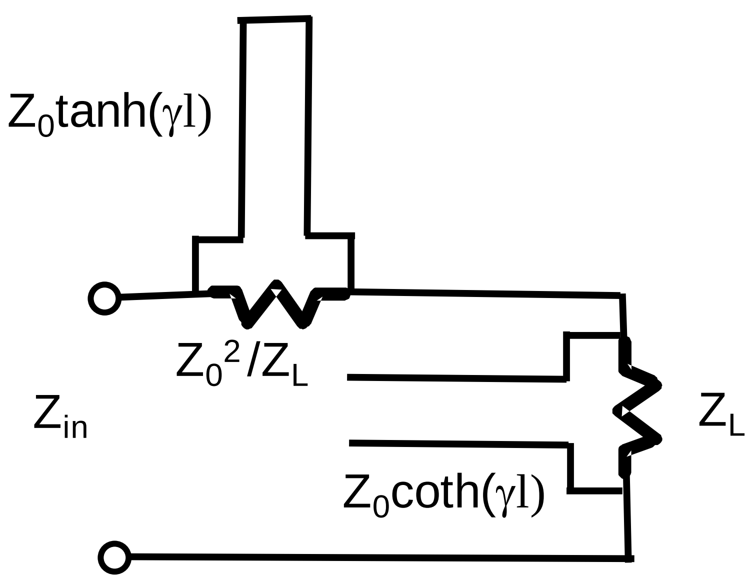

Figure 2. New equivalent circuit of Fig. 1 Z_{in}, using only lumped impedances and open and shorted stubs.

Now the important features of the arbitrarily loaded transmission line are clear from Fig. 2, as long as we remember what open and shorted stubs do. Recall that at quarter-wave frequency (also any odd number of quarter-waves), an open stub becomes a short, and a shorted stub becomes open. This tells us immediately that at an odd number of quarter-waves, Z_L disappears and the "inversion impedance" Z_0^2/Z_L emerges. Also, at zero frequency or at an even number of quarter waves, Z_L is restored and the inversion impedance disappears. The circuit is also correct for intermediate frequencies and for transients, and gives one a feel for what happens off-resonance, or in time domain, without using a computer. Finally, the well-known matching impedance problem is readily solved, i.e., for Z_{in} to match to a source impedance Z_s given Z_L, choose a quarter wavelength line section (resulting in the inversion impedance) and therefore choose $Z_0 = \sqrt{Z_L Z_s}$.

The equivalent Yin formulation is found by taking the reciprocal of (1) and dividing top and bottom by Z_0Z_L. Recognizing the Y (admittance) quantities as the reciprocal of the Z (impedance) quantities, we now have

$$Y_{in} = Y_0 \left[\frac{Y_L \cosh(\gamma l) + Y_0 \sinh(\gamma l)}{Y_0 \cosh(\gamma l) + Y_L \sinh(\gamma l)} \right] \quad . \quad (6)$$

This is exactly analogous to (1), so we can perform the same manipulations (2) through (4) and arrive at an expression analogous to (5), or

$$Y_{in} = \left[\frac{Y_0^{\ 2}}{Y_L} \right] \Bigg\| Y_0 \tanh(\gamma l) + Y_L \, \| Y_0 \coth(\gamma l) \quad . \quad (7)$$

For admittances, the "parallel" symbol now denotes a series circuit, the tanh and coth elements denote open circuit and short circuit lines, respectively, and added admittances occur with parallel circuits. Thus the equivalent circuit is complementary to Fig. 2, a parallel combination of series elements as in Figure 3a (Y form) and Figure 3b (Z form).

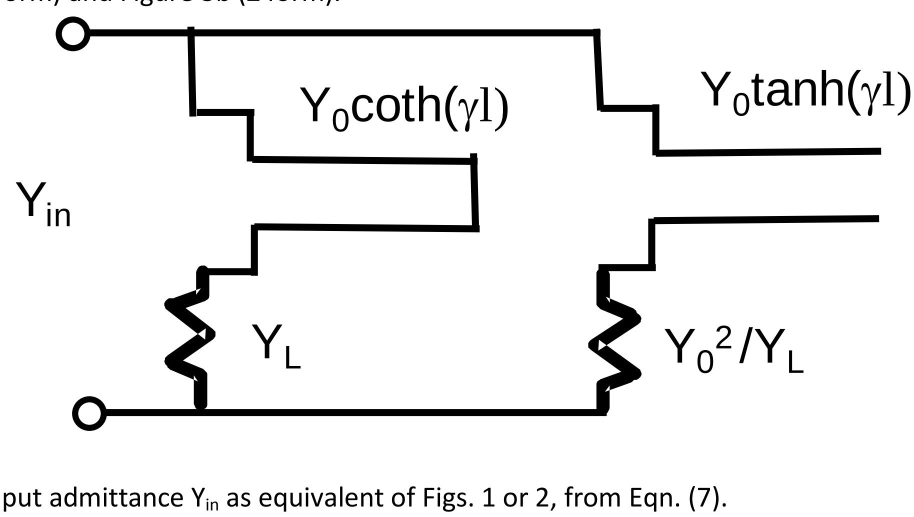

Figure 3a. Input admittance Y_{in} as equivalent of Figs. 1 or 2, from Eqn. (7).

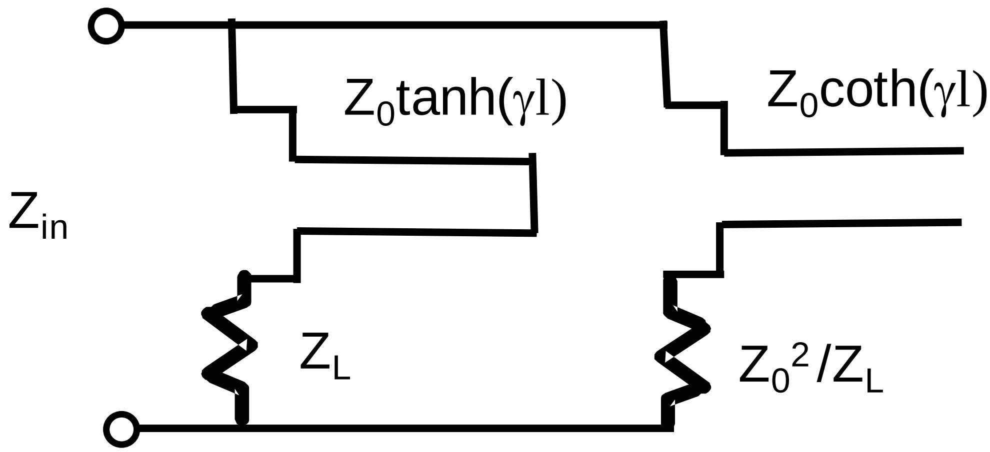

Figure 3b. Input impedance Z_{in} in parallel-series form, from Eqn. (7), equivalent to Figs. 1-3a.

This picture of the impedance Z_{in} or admittance Y_{in} of an arbitrarily terminated transmission line does not ordinarily appear in the related engineering textbooks, yet it lucidly illustrates the major features of unmatched line impedance that we all strive to recall and use. I hope that readers will find it useful.

References:

[1] S. Ramo, J. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics* (New York: John Wiley & Sons, 1965).
[2] G. Matthaei, L. Young, and E.M.T. Jones, *Microwave Filters, Impedance-Matching Networks, and Coupling Structures* (New York: McGraw-Hill, 1964; reprinted by Artech House, 1980).
[3] N. Marcuvitz, *Waveguide Handbook* (vol. 10, MIT Radiation Lab series; McGraw-Hill, 1951).

TIMOTHY J. MALONEY
timothy.j.maloney@intel.com

Timothy J. Maloney received an S.B. degree in physics from the Massachusetts Institute of Technology in 1971, an M.S. in physics from Cornell University in 1973, and a Ph.D. in electrical engineering from Cornell in 1976, where he was a National Science Foundation Fellow. He was a Postdoctoral Associate at Cornell until 1977, when he joined the Central Research Laboratory of Varian Associates, Palo Alto, CA. At Varian until 1984, he worked on III-V semiconductor photocathodes, solar cells and microwave devices, as well as silicon molecular beam epitaxy and MOS process technology. Since 1984 he has been with Intel Corp., Santa Clara, CA, where he has been concerned with integrated circuit ESD protection, CMOS latchup testing, fab process reliability, signal integrity, system ESD testing, and design and testing of standard IC layouts. He is now a Senior Principal Engineer at Intel. He has received the Intel Achievement Award for his patented ESD protection devices, which have achieved breakthrough ESD performance enhancements for a wide variety of Intel products. He now holds thirty-seven patents, with several more pending.

Dr. Maloney received Best Paper Awards for his contributions to the EOS/ESD Symposium in 1986 and 1990, was General Chairman for the 1992 EOS/ESD Symposium, and received the ESD

Association's Outstanding Contributions Award in 1995. He has taught short courses at UCLA, University of Wisconsin, and UC Berkeley. He is co-author of a book, "Basic ESD and I/O Design" (Wiley, 1998), and is a Fellow of the IEEE.